

MODERATE DEVIATION PRINCIPLE FOR EXPONENTIALLY ERGODIC MARKOV CHAIN

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ABSTRACT. For $\frac{1}{2} < \alpha < 1$, we propose the MDP analysis for family

$$S_n^\alpha = \frac{1}{n^\alpha} \sum_{i=1}^n H(X_{i-1}), \quad n \geq 1,$$

where $(X_n)_{n \geq 0}$ be a homogeneous ergodic Markov chain, $X_n \in \mathbb{R}^d$, when the spectrum of operator P_x is continuous. The vector-valued function H is not assumed to be bounded but the Lipschitz continuity of H is required. The main helpful tools in our approach are Poisson equation and Stochastic Exponential; the first enables to replace the original family by $\frac{1}{n^\alpha} M_n$ with a martingale M_n while the second to avoid the direct Laplace transform analysis.

1. INTRODUCTION AND DISCUSSION

Let $(X_n)_{n \geq 0}$ be a homogeneous ergodic Markov chain, $X_n \in \mathbb{R}^d$ with the transition probability kernel for n steps: $P_x^{(n)} = P^{(n)}(x, dy)$ (for brevity $P_x^{(1)} := P_x$) and the unique invariant measure μ .

Let H be a measurable function $\mathbb{R}^d \xrightarrow{H} \mathbb{R}^p$ with $\int_{\mathbb{R}^d} |H(z)| \mu(dz) < \infty$ and

$$\int_{\mathbb{R}^d} H(z) \mu(dz) = 0. \quad (1.1)$$

Set

$$S_n^\alpha = \frac{1}{n^\alpha} \sum_{i=1}^n H(X_{i-1}), \quad n \geq 1; \quad (0.5 < \alpha < 1).$$

In this paper, we examine the moderate deviation principle (MDP) for families $(S_n^\alpha)_{n \geq 1}$ when the spectrum of operator P_x is continuous.

It is well known that for bounded H satisfying (1.1) ((H) - condition), the most MDP compatible Markov chains are characterized by *eigenvalues gap condition* (EG) (see Wu, [16], [17], Gong and Wu, [7], and citations therein):

the unit is an isolated, simple and the only eigenvalue with modulus 1 of the transition probability kernel P_x .

In the framework of (H)-(EG) conditions, the MDP is valid with the rate of speed $n^{-(2\alpha-1)}$ and the rate function $I(y), y \in \mathbb{R}^d$

$$I(y) = \begin{cases} \frac{1}{2} \|y\|_{B^\oplus}^2, & B^\oplus B y = y \\ \infty, & \text{otherwise,} \end{cases} \quad (1.2)$$

where B^\oplus is the pseudoinverse matrix (in Moore-Penrose sense, see e.g.[1]) for the matrix

$$B = \int_{\mathbb{R}^d} H(x)H^*(x)\mu(dx) + \sum_{n \geq 1} \int_{\mathbb{R}^d} \left[H(x)(P_x^{(n)}H)^* + (P_x^{(n)}H)H^*(x) \right] \mu(dx) \quad (1.3)$$

(henceforth, $*$, $|\cdot|$, and $\|\cdot\|_Q$ are the transposition symbol, \mathbb{L}^1 norm and \mathbb{L}^2 norm with the kernel Q ($\|x\|_Q = \sqrt{\langle x, Qx \rangle}$) respectively).

Owing to the quadratic form rate function, the MDP is an attractive tool for an asymptotic analysis in many areas, say, with thesis

“MDP instead of CLT”

In this paper, we intend to apply the MDP analysis to Markov chain defined by the recurrent equation

$$X_n = f(X_{n-1}, \xi_n), \quad n \geq 1$$

generated by i.i.d. sequence $(\xi_n)_{n \geq 1}$ of random vectors, where f is some vector-valued measurable function. Obviously, the function f and the distribution of ξ_1 might be specified in this way P_x satisfies (EG). For instance, if $d = 1$ and

$$X_n = f(X_{n-1}) + \xi_n,$$

then for bounded f and Laplacian random variable ξ_1 (EG) holds. However, (EG) fails for many useful in applications ergodic Markov chains. For $d = 1$, a typical example is Gaussian Markov chain defined by the linear recurrent equation governed by i.i.d. sequence of $(0,1)$ -Gaussian random variables (here $|a| < 1$)

$$X_n = aX_{n-1} + \xi_n.$$

In this paper, we avoid a verification of (EG). Although our approach is close to conceptions of “Multiplicative Ergodicity” (see Balaji and Myen [2]) and “Geometrical Ergodicity” (see Kontoyiannis and Meyn, [8] and Meyn and Tweedie, [11]), Chen and Guillin, [4]) we do not follow explicitly these methodologies.

Our main tools are the Poisson equation and the Puhalskii theorem from [15]. The Poisson equation permits to reduce the MDP verification for $(S_n^\alpha)_{n \geq 1}$ to $(\frac{1}{n^\alpha}M_n)_{n \geq 1}$, where M_n is a martingale generated by Markov chain, while the Puhalskii theorem allows to replace an asymptotic analysis for the Laplace transform of $\frac{1}{n^\alpha}M_n$ by the asymptotic analysis for, so called, *Stochastic Exponential*

$$\mathcal{E}_n(\lambda) = \prod_{i=1}^n E\left(\exp\left[\left\langle \lambda, \frac{1}{n^\alpha}(M_i - M_{i-1}) \right\rangle\right] \middle| X_{i-1}\right), \quad \lambda \in \mathbb{R}^d \quad (1.4)$$

being the product of the conditional Laplace transforms for martingale increments.

An effectiveness of the Poisson equation approach (method of corrector) combined with the stochastic exponential is well known from the proofs of functional central limit theorem (FCLT) for the family $(S_n^{0.5})_{n \geq 1}$ (see,

e.g. Papanicolaou, Stroock and Varadhan [12], Ethier and Kurtz [6], Bhattacharya [3], Pardoux and Veretennikov [13]) and with $0.5 < \alpha < 1$ for the MDP analysis for the continuous time case (see e.g. [9], [10]).

2. FORMULATION OF MAIN RESULT

We consider Markov chain $(X_n)_{n \geq 0}$, $X_n \in \mathbb{R}^d$ defined by a nonlinear recurrent equation

$$X_n = f(X_{n-1}, \xi_n), \quad (2.1)$$

where $f = f(z, v)$ is a vector function with entries $f_1(z, v), \dots, f_d(z, v)$, $u \in \mathbb{R}^d$, $v \in \mathbb{R}^p$ and $(\xi_n)_{n \geq 1}$ is i.i.d. sequence of random vectors of the size p .

We fix the following assumptions.

Assumption 2.1. *Entries of f are Lipschitz continuous functions in the following sense:*

$$\begin{aligned} & |f_i(z_1 \dots, z_{j-1}, z'_j, z_{j+1} \dots, z_d, v_1, \dots, v_p) \\ & \quad - f_i(z_1 \dots, z_{j-1}, z''_j, z_{j+1} \dots, z_d, v_1, \dots, v_p)| \\ & \leq \varrho_{ij} |z'_j - z''_j|, \end{aligned}$$

$$|f(z, v') - f(z, v'')| \leq \varrho |v' - v''|,$$

where

$$\max_{i,j} \varrho_{ij} = \varrho < 1.$$

Assumption 2.2. *For sufficiently small positive δ ,*

$$E e^{\delta |\xi_1|} < \infty.$$

Theorem 2.1. *Under Assumptions 2.1 and 2.2, the Markov chain is ergodic with the invariant measure μ such that $\int_{\mathbb{R}^d} |z| \mu(dz) < \infty$. For any Lipschitz continuous function H with $\int_{\mathbb{R}^d} H(z) \mu(dz) = 0$, the family $(S_n^\alpha)_{n \geq 1}$ obeys the MDP in the metric space (\mathbb{R}^d, r) (r is the Euclidean metric) with the rate of speed $n^{-(2\alpha-1)}$ and the rate function given in (1.2).*

Remark 1. *The assumptions of Theorem 2.1 do not guarantee (EG). The function H satisfies the linear growth condition. Moreover, an existence of the continuous component of ξ_1 -distribution, as in is not to be assumed.*

Consider now a linear recurrent equation

$$X_n = AX_{n-1} + \xi_n$$

governed by i.i.d. sequence $(\xi_n)_{n \geq 1}$ of random vectors, where A is matrix of the size $d \times d$ and entries A_{ij} . Now, Assumption 2.1 reads as: $\max_{ij} |A_{ij}| < 1$. This assumption is very restrictive and we replace it by more natural one

Assumption 2.3. *The eigenvalues of A lie within the unit circle.*

Theorem 2.2. *Under Assumption 2.3, the Markov chain is ergodic with the invariant measure μ such that $\int_{\mathbb{R}^d} \|z\|^2 \mu(dz) < \infty$. For any Lipschitz continuous function H with $\int_{\mathbb{R}^d} H(z) \mu(dz) < \infty$, the family $(S_n^\alpha)_{n \geq 1}$ obeys the MDP in the metric space (\mathbb{R}^d, r) with the rate of speed $n^{-(2\alpha-1)}$ and the rate function given in (1.2).*

3. PRELIMINARIES

3.1. (EG)-(H) conditions. To clarify our approach to the MDP analysis, let us demonstrate its application for (EG)-(H) setting.

The (EG) condition provides the geometric ergodicity of $P_x^{(n)}$ to the invariant measure μ in the total variation norm: there exist constants $K > 0$ and $\varrho \in (0, 1)$ such that for any $x \in \mathbb{R}^d$

$$\|P_x^{(n)} - \mu\|_{\text{tv}} \leq K \varrho^n, \quad n \geq 1.$$

So, (EG)-(H) conditions provide the existence of bounded function

$$U(x) = H(x) + \sum_{n \geq 1} P_x^{(n)} H \quad (3.1)$$

which solves the Poisson equation

$$H(x) = H(x) + P_x U. \quad (3.2)$$

In view of the Markov property, $\zeta_i := U(X_i) - P_{X_{i-1}} U$, $i \geq 1$ is the sequence of bounded martingale-differences with respect to the filtration generated by Markov chain. Hence, $M_n = \sum_{i=1}^n \zeta_i$ is the martingale with bounded increments. With the help of Poisson's equation we get the following decomposition

$$\sum_{i=1}^n H(X_{i-1}) = \underbrace{U(x) - U(X_n)}_{\text{corrector}} + M_n. \quad (3.3)$$

Owing to the boundedness of U , the families S_n^α and $\frac{1}{n^\alpha} M_n$ share the same MDP. This fact enables us to verify the MDP for $(\frac{1}{n^\alpha} M_n)_{n \geq 1}$.

Assume for a moment that ζ_i 's are also independent and identically distributed random vectors. Recall that $E\zeta_1 = 0$ and denote $B = E\zeta_1 \zeta_1^*$. Introduce the Laplace transform for $\frac{1}{n^\alpha} M_n$:

$$\mathcal{E}_n(\lambda) = \left(E e^{\langle \lambda, \frac{\zeta_1}{n^\alpha} \rangle} \right)^n, \quad \lambda \in \mathbb{R}^d. \quad (3.4)$$

It is well known that the MDP for $\frac{1}{n^\alpha} M_n$ is provided by the following conditions: B is not singular matrix and

$$\lim_{n \rightarrow \infty} n^{2\alpha-1} \log \mathcal{E}_n(\lambda) = \frac{1}{2} \langle \lambda, B \lambda \rangle, \quad \lambda \in \mathbb{R}^d.$$

The framework of this proof might be adapted to the case considered in the paper. Instead of B , we introduce the conditional covariance matrix $E(\zeta_i \zeta_i^* | X_{i-1})$ and, instead of the Laplace transform (3.4), the stochastic exponential (1.4), having a form

$$\mathcal{E}_n(\lambda) = \prod_{i=1}^n E \left(e^{\langle \lambda, \frac{\zeta_i}{n^\alpha} \rangle} | X_{i-1} \right), \quad \lambda \in \mathbb{R},$$

which is not the Laplace transform itself. The homogeneity of Markov chain and the definition of ζ_i provide $E(\zeta_i \zeta_i^* | X_{i-1}) = B(X_{i-1})$ for the matrix-valued function

$$B(x) = P_x U U^* - P_x U (P_x U)^*. \quad (3.5)$$

The Poisson equation (3.2) and its solution (3.1) allow to transform (3.5) into

$$B(x) = H(x)H^*(x) + \sum_{n \geq 1} \left[H(x)(P_x^{(n)} H)^* + (P_x^{(n)} H)H^* \right],$$

that is for B defined in (1.3)

$$\int_{\mathbb{R}^d} B(z) \mu(dz) = B.$$

Now, we are in the position to formulate

Puhalskii Theorem. *Assume B from (1.3) is nonsingular matrix and for any $\varepsilon > 0$, $\lambda \in \mathbb{R}^d$*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P \left(\left| n^{2\alpha-1} \log \mathcal{E}_n(\lambda) - \frac{1}{2} \langle \lambda, B \lambda \rangle \right| > \varepsilon \right) = -\infty. \quad (3.6)$$

Then, the family $\frac{1}{n^\alpha} M_n$, $n \geq 1$ possesses the MDP in the metric space (\mathbb{R}^d, r) (r is the Euclidean metric) with the rate of speed $n^{-(2\alpha-1)}$ and rate function $I(y) = \frac{1}{2} \|y\|_{B^{-1}}^2$.

Remark 2. *The condition (3.6) is provided by*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P \left(\frac{1}{n} \left| \sum_{i=1}^n \langle \lambda, [B(X_{i-1}) - B] \lambda \rangle \right| > \varepsilon \right) &= -\infty \\ \lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P \left(\frac{1}{6n^{1+\alpha}} \sum_{i=1}^n E \left[|\zeta_i|^3 e^{n^{-\alpha} |\zeta_i|} | X_{i-1} \right] > \varepsilon \right) &= -\infty. \end{aligned} \quad (3.7)$$

The second condition in (3.7) is implied by the boundedness of $|\zeta_i|$'s. The first part in (3.7) is known as Dembo's conditions, [5], formulated as follows: for any $\varepsilon > 0$, $\lambda \in \mathbb{R}^d$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log P \left(\frac{1}{n} \left| \sum_{i=1}^n \langle \lambda, [B(X_{i-1}) - B] \lambda \rangle \right| > \varepsilon \right) < 0.$$

In order to verify the first condition in (3.7), we will follow to the framework of Poisson's equation technique. We introduce the function

$$h(x) = \langle \lambda, [B(x) - B] \lambda \rangle$$

which is bounded and

$$\int_{\mathbb{R}^d} h(z) \mu(dz) = 0.$$

Then the function $u(x) = h(x) + \sum_{n \geq 1} P_x^{(n)} h$ is well defined and solves the Poisson equation $u(x) = h(x) + P_x u$. Similarly to (3.3), we have

$$\frac{1}{n} \sum_{i=1}^n h(X_{i-1}) = \frac{u(x) - u(X_n)}{n} + \frac{m_n}{n},$$

where $m_n = \sum_{i=1}^n z_i$ is the martingale with bounded martingale-differences $(z_i)_{i \geq 1}$. Since u is bounded, the first condition in (3.7) is reduced to

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(|m_n| > n\varepsilon) = -\infty \quad (3.8)$$

while (3.8) is provided by Theorem A.1 in Appendix which states that (3.8) holds for any martingale with bounded increments.

3.1.1. Singular B . Though to the conditions from (3.7) remain to hold when B is singular, the Puhalskii theorem is no longer valid. Nevertheless, we shall use this theorem as an auxiliary tool.

It is well known that the family $\frac{M_n}{n^\alpha}$, $n \geq 1$ obeys the MDP with the rate of speed $n^{-(2\alpha-1)}$ and the rate function given in (1.2) provided that

$$\begin{aligned} \overline{\lim}_{C \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{M_n}{n^\alpha}\right\| > C\right) &= -\infty \\ \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{M_n}{n^\alpha} - y\right\| \leq \varepsilon\right) &\leq -I(y) \\ \underline{\lim}_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{M_n}{n^\alpha} - y\right\| \leq \varepsilon\right) &\geq -I(y). \end{aligned} \quad (3.9)$$

The first condition in (3.9) provides the exponential tightness in the metric r while the next others the local MDP.

In order to verify of (3.9), we introduce “regularized” family $\frac{M_n^\beta}{n^\alpha}$, $n \geq 1$ with

$$M_n^\beta = M_n + \sqrt{\beta} \sum_{i=1}^n \theta_i,$$

where β is a positive parameter and $(\theta_i)_{i \geq 1}$ is a sequence of zero mean i.i.d. Gaussian random vectors with $\text{cov}(\theta_1, \theta_1) =: \mathbf{I}$ (\mathbf{I} is the unite matrix). The Markov chain and $(\theta_i)_{i \geq 1}$ are assumed to be independent objects.

It is clear that for this setting the matrix B is transformed into the positive definite matrix $B_\beta = B + \beta \mathbf{I}$. Now, the Puhalskii theorem is applicable and guarantees the MDP with the same rate of speed and the rate function $I_\beta(y) = \frac{1}{2} \|y\|_{B_\beta^{-1}}^2$. As the corollary of the MDP, $\frac{M_n^\beta}{n^\alpha}$, $n \geq 1$ is exponentially tight (see [14]) and obeys the local MDP:

$$\begin{aligned} \overline{\lim}_{C \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{M_n^\beta}{n^\alpha}\right\| > C\right) &= -\infty \\ \overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{M_n^\beta}{n^\alpha} - u\right\| \leq \varepsilon\right) &\leq -I_\beta(y) \\ \underline{\lim}_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{M_n^\beta}{n^\alpha} - u\right\| \leq \varepsilon\right) &\geq -I_\beta(y). \end{aligned} \quad (3.10)$$

Notice now that (3.9) is implied by (3.10) provided that

$$\lim_{\beta \rightarrow 0} I_\beta(y) = \begin{cases} \frac{1}{2} \|u\|_{B^\oplus}^2, & B^\oplus B y = y \\ \infty, & \text{otherwise} \end{cases} \quad (3.11)$$

and

$$\lim_{\beta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} P\left(\left\|\frac{\sqrt{\beta}}{n^\alpha} \sum_{i=1}^n \theta_i\right\| > \eta\right) = -\infty, \quad \forall \eta > 0. \quad (3.12)$$

Let T be the orthogonal matrix transforming B to the diagonal form: $\text{diag}(B) = T^*BT$. Then, owing to

$$2I_\beta(y) = y^*(\beta I + B)^{-1}y = y^*T(\beta I + \text{diag}(B))^{-1}T^*y,$$

for $y = B^\oplus By$ we have (recall that $B^\oplus BB^\oplus = B^\oplus$, see [1])

$$\begin{aligned} 2I_\beta(y) &= y^*B^\oplus BT(\beta I + \text{diag}(B))^{-1}T^*y \\ &= y^*B^\oplus TT^*BT(\beta I + \text{diag}(B))^{-1}T^*y \\ &= y^*B^\oplus T \text{diag}(B)(\beta I + \text{diag}(B))^{-1}T^*y \\ &\xrightarrow{\beta \rightarrow 0} y^*B^\oplus T \text{diag}(B) \overset{\oplus}{\text{diag}(B)} T^*y \\ &= y^*B^\oplus T \text{diag}(B) T^*T \overset{\oplus}{\text{diag}(B)} T^*y \\ &= y^*B^\oplus BB^\oplus u = u^*B^\oplus y = \|y\|_{B^\oplus}^2 = 2I(y). \end{aligned}$$

If $y \neq B^\oplus By$, $\lim_{\beta \rightarrow 0} 2I_\beta(y) = \infty$.

Thus, (3.11) holds true.

Since $(\theta_i)_{i \geq 1}$ is i.i.d. sequence of random vectors and entries of ξ_1 are i.i.d. $(0, 1)$ -Gaussian random variables, the verification of (3.12) is reduced to

$$\lim_{\beta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P\left(\left|\sum_{i=1}^n \vartheta_i\right| > \frac{n^\alpha \eta}{\sqrt{\beta}}\right) = -\infty, \quad (3.13)$$

where $(\vartheta_i)_{i \geq 1}$ is a sequence of i.i.d. $(0, 1)$ -Gaussian random variables. It is clear that (3.13) is equivalent to

$$\lim_{\beta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P\left(\pm \sum_{i=1}^n \vartheta_i > \frac{n^\alpha \eta}{\sqrt{\beta}}\right) = -\infty$$

and, moreover, it suffices to establish “+” only.

By the Chernoff inequality with $\lambda > 0$, we find that

$$P\left(\sum_{i=1}^n \theta_i > \frac{n^\alpha \eta}{\sqrt{\beta}}\right) \leq \exp\left(-\lambda \frac{n^\alpha \eta}{\sqrt{\beta}} + n \frac{\lambda^2}{2}\right)$$

while the choice of $\lambda = \frac{n^\alpha \eta}{n\sqrt{\beta}}$ provides

$$\frac{1}{n^{2\alpha-1}} \log P\left(\sum_{i=1}^n \eta_i > \frac{n^\alpha \eta}{\sqrt{\beta}}\right) \leq -\frac{\eta^2}{2\beta} \xrightarrow{\beta \rightarrow 0} -\infty.$$

3.2. Virtual scenario.

- (EG)-(H) are not assumed
- the ergodicity of Markov is checked
- H is chosen to hold (1.1).

(1) We assume (3.1). Then the function U solves the Poisson equation and the decomposition from (3.3) is valid with $M_n = \sum_{i=1}^n \zeta_i$, where $\zeta_i = u(X_i) - P_{X_{i-1}}u$. We assume that

$$E\zeta_i^* \zeta_i \leq \text{const.}$$

$$E\left[|\zeta_i|^3 e^{n^{-\alpha}|\zeta_i|} | X_{i-1}\right] \leq \text{const.}$$

(2) With $B(x)$ and B are defined in (3.5) and (1.3) respectively, set

$$h(x) = \left\langle \lambda, [B(x) - B]\lambda \right\rangle, \quad \lambda \in \mathbb{R}^d.$$

We assume that

- (i) $u(x) = h(x) + \sum_{n \geq 1} P_x^{(n)}h$ is well defined
- (ii) for $z_i = u(X_i) - P_{X_{i-1}}u$,

$$Ez_i^2 \leq \text{const.}$$

$$E\left[|z_i|^3 e^{n^{-\alpha}|z_i|} | X_{i-1}\right] \leq \text{const.}$$

(3) We assume that for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(|U(X_n)| > n^\alpha \varepsilon) = -\infty$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(|u(X_n)| > n^\alpha \varepsilon) = -\infty.$$

Notice that (EG)-(H) provide (1)-(3) and even if (EG)-(H) fail, (1)-(3) may fulfill. Moreover, (1)-(3) guarantee the validity for all steps of the proof given in Section 3.1, so that if the ergodic property of Markov chain hold, (1)-(3) provides the MDP.

We use this scenario for the proofs of Theorems 2.1 and 2.2.

4. THE PROOF OF THEOREM 2.1

Here, we follow the virtual scenario from Section 3.2.

4.1. Ergodic property.

Lemma 4.1. *Under Assumption 2.1, $(X_n)_{n \geq 0}$ possesses the unique probability invariant measure μ with $\int_{\mathbb{R}^d} |z| \mu(dz) < \infty$.*

Proof. Let us initialize the recursion given in (2.1) by a random vector X_0 , independent of $(\xi_n)_{n \geq 1}$, with the distribution function ν such that $\int_{\mathbb{R}^d} |x| \nu(dx) < \infty$ and denote $\mu^n(dz) = \int_{\mathbb{R}^d} P_x^{(n)}(dz) \nu(dx)$. We show that the family $(\mu^n)_{n \geq 1}$ is tight in the Levy-Prohorov metric:

$$\lim_{k \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \mu^n(|z| > k) = 0.$$

Since by the Chebyshev inequality $\mu^n(|z| > k) \leq \frac{E|X_n|}{k}$, it suffices to show that

$$\sup_{n \geq 1} E|X_n| < \infty. \tag{4.1}$$

By Assumption 2.1, we have

$$\begin{aligned} |X_n| &= |f(0, \xi_n) + (f(X_{n-1}, \xi_n) - f(0, \xi_n))| \\ &\leq |f(0, \xi_n)| + |f(X_{n-1}, \xi_n) - f(0, \xi_n)| \\ &\leq |f(0, \xi_n)| + \varrho |X_{n-1}| \\ &\leq |f(0, 0)| + \ell |\xi_n| + \varrho |X_{n-1}|. \end{aligned}$$

Hence, $E|X_n| \leq |f(0, 0)| + \ell E|\xi_1| + \varrho E|X_{n-1}|$. Since $E|X_0| < \infty$, we have $E|X_n| < \infty$ and, moreover,

$$E|X_n| \leq E|X_0| + \frac{|f(0, 0)| + \ell E|\xi_1|}{1 - \varrho}, \quad n \geq 1, \quad (4.2)$$

i.e. (4.1) holds true.

Further, by the Prohorov theorem, the sequence of μ^n , $n \nearrow \infty$, contains further subsequence $\mu^{n'}$, $n' \nearrow \infty$ converging in the Levy-Prohorov metric to a limit μ being the probability measure on \mathbb{R}^d . In other words, for any bounded and continuous function g

$$\lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} g(z) \mu^{n'}(dz) = \int_{\mathbb{R}^d} g(z) \mu(dz).$$

With $L > 0$, taking $g(z) = L \wedge |z|$, we obtain

$$\int_{\mathbb{R}^d} (L \wedge |z|) \mu(dz) = \lim_{n' \rightarrow \infty} E(L \wedge |X_{n'}|) \leq \overline{\lim}_{n \rightarrow \infty} E|X_n| \leq \text{const.}$$

(see (4.2)). Then, by the monotone convergence theorem, it holds that

$$\int_{\mathbb{R}^d} |z| \mu(dz) \leq \overline{\lim}_{n \rightarrow \infty} E|X_n| < \infty.$$

Now, we show that μ is an invariant measure of the Markov chain, that is for any nonnegative, bounded and measurable function g

$$\int_{\mathbb{R}^d} g(x) \mu(dx) = \int_{\mathbb{R}^d} P_x g \mu(dx). \quad (4.3)$$

It suffices to verify (4.3) for continuous function only. The general statement is obtained by a monotonic type approximation.

For notational convenience, write X_n^x and X_n^ν , if $X_0 = x$ and X_0 is distributed in the accordance with ν . Making use Assumption 2.1, we find that

$$|X_n^x - X_n^\nu| \leq \varrho |X_{n-1}^x - X_{n-1}^\nu|, \quad n \geq 1$$

and claim that $|X_n^x - X_n^\nu|$ converges to zero exponentially fast as long as $n \rightarrow \infty$. For any $x \in \mathbb{R}^d$, the latter provides

$$\lim_{n' \rightarrow \infty} E g(X_{n'}^x) = \int_{\mathbb{R}^d} g(x) \mu(dx).$$

This and the fact that $(X_n)_{n \geq 0}$ is the homogeneous Markov chain also imply

$$\lim_{n' \rightarrow \infty} E g(X_{n'+1}^x) = \int_{\mathbb{R}^d} g(z) \mu(dz).$$

On the other hand, taking into the consideration $E g(X_{n'+1}^x) = E P_{X_n^x} g$, the above relation is nothing but $\lim_{n' \rightarrow \infty} E P_{X_n^x} g = \int_{\mathbb{R}^d} g(z) \mu(dz)$. The next key tool in the proof is the Feller property: for any bounded and continuous g ,

the function $P_x g = Eg(f(x, \xi_1))$ is the continuous as well. So, by the Feller property $\lim_{n' \rightarrow \infty} EP_{X_{n'}} g = \int_{\mathbb{R}^d} P_x g \mu(dx)$.

Thus, (4.3) holds.

Assume μ' is another invariant probability measure, $\mu' \neq \mu$. Then taking two random vectors X_0^μ and $X_0^{\mu'}$, distributed in the accordance to μ and μ' respectively and independent of $(\xi_n)_{n \geq 1}$, we create two stationary Markov chains (X_n^μ) and $(X_n^{\mu'})$ defined on the same probability space as:

$$\begin{aligned} X_n^\mu &= f(X_{n-1}^\mu, \xi_n) \\ X_n^{\mu'} &= f(X_{n-1}^{\mu'}, \xi_n). \end{aligned}$$

By Assumption 2.1, $|X_n^\mu - X_n^{\mu'}| \leq \varrho |X_{n-1}^\mu - X_{n-1}^{\mu'}|$, i.e. $\lim_{n \rightarrow \infty} |X_n^\mu - X_n^{\mu'}| = 0$ what contradicts $\mu \neq \mu'$. \square

4.2. The verification of (1). Let K be the Lipschitz constant for H . Then $|H(x)| \leq |H(0)| + K|x|$ and $\int_{\mathbb{R}^d} |H(z)| \mu(dz) < \infty$. Hence, (1.1) is the correct assumption and for the stationary Markov chain X_n^μ we have $EH(X_n^\mu) \equiv 0$. Then,

$$\begin{aligned} |EH(X_n^x)| &= |E(H(X_n^x) - H(X_n^\mu))| \\ &\leq K \varrho^n E|x - X_n^\mu| \leq K(1 + |x|) \varrho^n. \end{aligned}$$

Therefore, $\sum_{n \geq 1} |EH(X_n^x)| \leq \frac{K}{1-\varrho}(1 + |x|)$. Consequently, the function $U(x)$, given in (3.1), is well defined and solves the Poisson equation.

Recall that $\zeta_i = U(X_i) - P_{X_{i-1}} U$.

Lemma 4.2. *The function $U(x)$ possesses the following properties:*

- 1) $U(x)$ is Lipschitz continuous;
- 2) $P_x(UU^*) - P_x U(P_x U)^*$ is bounded and Lipschitz continuous;
- 3) For sufficiently small $\delta > 0$ and any $i \geq 1$

$$E\left(|U(X_i) - P_{X_{i-1}} U|^3 e^{\delta|U(X_i) - P_{X_{i-1}} U|} |X_{i-1}\right) \leq \text{const.}$$

Proof. 1) Since by Assumption 2.1,

$$|X_n^{x'} - X_n^{x''}| \leq \varrho |X_{n-1}^{x'} - X_{n-1}^{x''}|, \quad |X_0^{x'} - X_0^{x''}| \leq |x' - x''|,$$

we have

$$\begin{aligned} |U(x') - U(x'')| &\leq |H(x') - H(x'')| + \sum_{n \geq 1} E|H(X_n^{x'}) - H(X_n^{x''})| \\ &\leq \frac{K}{1-\varrho} |x' - x''|. \end{aligned} \tag{4.4}$$

2) Recall (see (3.5))

$$P_x(UU^*) - P_x U(P_x U)^* = B(x)$$

and denote $B_{pq}(x)$, $p, q = 1, \dots, d$ the entries of matrix $G(x)$. Also, denote by $U_p(x)$, $p = 1, \dots, d$ the entries of $U(x)$. Since $B(x)$ is nonnegative definite matrix, it suffices to show only that $B_{pp}(x)$'s are bounded functions. Denote

$F(z)$ the distribution function of ξ_1 . Taking into the consideration (4.4) and Assumption 2.1, we get

$$\begin{aligned} B_{pp}(x) &= E\left(U_p(f(x, \xi_1)) - \int_{\mathbb{R}^d} U_p(f(x, z)) dF(z)\right)^2 \\ &\leq \frac{(K\ell)^2}{(1-\varrho)^2} E\left|\int_{\mathbb{R}^d} |\xi_1 - z| dF(z)\right|^2 \leq 4 \frac{(K\ell)^2}{(1-\varrho)^2} E|\xi_1|^2 < \infty. \end{aligned}$$

The Lipschitz continuity of $B_{pq}(x)$ is proved similarly. Write

$$B_{pq}(x') - B_{pq}(x'') =: ab - cd,$$

where

$$\begin{aligned} a &= E\left(U_p(f(x', \xi_1)) - \int_{\mathbb{R}^d} U_q(f(x', z)) dF(z)\right) \\ b &= E\left(U_q(f(x', \xi_1)) - \int_{\mathbb{R}^d} U_q(f(x', z)) dF(z)\right) \\ c &= E\left(U_p(f(x'', \xi_1)) - \int_{\mathbb{R}^d} U_q(f(x'', z)) dF(z)\right) \\ d &= E\left(U_q(f(x'', \xi_1)) - \int_{\mathbb{R}^d} U_q(f(x'', z)) dF(z)\right). \end{aligned}$$

Now, applying $ab - cd = a(b - d) + d(a - c)$ and taking into account (4.4) and Assumption 2.1, we find that $|a|, |d| \leq \frac{2K\ell}{1-\varrho} E|\xi_1|$ and so

$$|B_{pq}(x') - B_{pq}(x'')| \leq \frac{4K^2\ell\varrho}{(1-\varrho)^2} E|\xi_1| |x' - x''|.$$

3) By (4.4) and Assumption 2.1

$$|U(X_i) - P_{X_{i-1}}U| \leq \frac{K\ell}{1-\varrho} (E|\xi_1| + |\xi_i|).$$

□

4.3. The verification of (2). The properties of $G(x)$ to be bounded and Lipschitz continuous provide the same properties for

$$h(x) = \langle \lambda, [B(x) - B]\lambda \rangle.$$

Hence (2) is provided by (1).

4.4. The verification of (3). Since U and u are Lipschitz continuous, they possess the linear growth condition, e.g. $|U(x)| \leq C(1 + |x|)$, $\exists C > 0$. So, (3) is reduced to the verification of

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(|X_n| > \varepsilon n^\alpha) = -\infty, \quad \varepsilon > 0. \quad (4.5)$$

Due to Assumption 2.1, we have

$$\begin{aligned} |X_n| &\leq |f(X_{n-1}, \xi_n)| \leq |f(0, \xi_n)| + \varrho|X_{n-1}| \\ &\leq |f(0, 0)| + \varrho|X_{n-1}| + \ell|\xi_n|. \end{aligned}$$

Iterating this inequality with $X_0 = x$ we obtain

$$\begin{aligned} |X_n| &\leq \varrho^n |x| + |f(0, 0)| \sum_{j=1}^n \varrho^{n-j} + \ell \sum_{j=1}^n \varrho^{n-j} |\xi_j| \\ &\leq |x| + \frac{|f(0, 0)|}{1 - \varrho} + \ell \sum_{j=0}^{n-1} \varrho^j |\xi_{n-j}|. \end{aligned}$$

Hence, (5.2) is reduced to

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P \left(\sum_{j=0}^{n-1} \varrho^j |\xi_{n-j}| \geq n^\alpha \varepsilon \right) = -\infty. \quad (4.6)$$

We verify (4.6) with the help of Chernoff's inequality: with δ , involving in Assumption 2.2, and $\gamma = \frac{\delta}{1-\varrho}$

$$P \left(\sum_{j=0}^{n-1} \varrho^j |\xi_{n-j}| \geq n^\alpha \varepsilon \right) \leq e^{-n^\alpha \gamma \varepsilon} E e^{\sum_{j=0}^{n-1} \gamma \varrho^j |\xi_{n-j}|}.$$

The i.i.d. property for ξ_j 's provides

$$E e^{\sum_{j=0}^{n-1} \gamma \varrho^j |\xi_{n-j}|} = E e^{\sum_{j=0}^{n-1} \gamma \varrho^j |\xi_1|} \leq E e^{\sum_{j=0}^{\infty} \gamma \varrho^j |\xi_1|} = E e^{\delta |\xi_1|} < \infty$$

and we get

$$\frac{1}{n^{2\alpha-1}} \log P \left(\sum_{j=0}^{n-1} \varrho^j |\xi_{n-j}| \geq n^\alpha \varepsilon \right) \leq -n^{1-\alpha} \delta \varepsilon + \frac{\log E e^{\delta |\xi_1|}}{n^{2\alpha-1}} \xrightarrow{n \rightarrow \infty} -\infty.$$

5. THE PROOF OF THEOREM 2.2

The proof of this theorem differs from the proof of Theorem 2.1 only in some details concerning to (L.1). So, only these parts of the proof are given below.

5.1. Ergodic property and invariant measure. Introduce $(\tilde{\xi}_n)_{n \geq 1}$ the independent copy of $(\xi_n)_{n \geq 1}$. Owing to

$$X_n = A^n x + \sum_{i=1}^n A^{n-i} \xi_i = A^n x + \sum_{i=0}^{n-1} A^i \xi_{n-i},$$

we introduce

$$\tilde{X}_n = A^n x + \sum_{i=0}^{n-1} A^i \tilde{\xi}_i \quad (5.1)$$

and notice that the i.i.d. property of $(\xi_i)_{i \geq 1}$ provides $(X_n)_{n \geq 0} \stackrel{\text{law}}{=} (\tilde{X}_n)_{n \geq 0}$.

By Assumption 2.3, $A^n \rightarrow 0$, $n \rightarrow \infty$, exponentially fast. Particularly,

$$\sum_{i=0}^{\infty} \text{trace} (A^i \text{cov}(\xi_1, \xi_1) (A^i)^*) < \infty,$$

so that $\lim_{n \rightarrow \infty} \tilde{X}_n = \sum_{i=0}^{\infty} A^i \tilde{\xi}_i$ a.s. and in \mathbb{L}^2 norm.

Thus, the invariant measure μ is generated by the distribution function of \tilde{X}_∞ . In addition, $E\|\tilde{X}_\infty\|^2 = \sum_{i=0}^{\infty} \text{trace}(A^i \text{cov}(\xi_1, \xi_1)(A^i)^*)$, so that

$$\int_{\mathbb{R}^d} \|z\|^2 \mu(dz) < \infty.$$

5.2. The verification of (1) and (2). Due to

$$(X_n^{x'} - X_n^{x''}) = A(X_{n-1}^{x'} - X_{n-1}^{x''}),$$

we have $(X_n^{x'} - X_n^{x''}) = A^n(x' - x'')$. Let us transform the matrix A into a Jordan form $A = TJT^{-1}$ and notice that $A^n = TJ^nT^{-1}$. It is well known that the maximal absolute value of entries of J^n is $n|\lambda|^n$, where $|\lambda|$ is the maximal absolute value among eigenvalues of A . By Assumption 2.3, $|\lambda| < 1$. So, there exist $K > 0$ and $\varrho < 1$ such that $|\lambda| < \varrho$. Then, entries A_{pq}^n of A^n are evaluated as: $|A_{pq}^n| \leq K\varrho^n$. Hence, $|X_n^{x'} - X_n^{x''}| \leq K\varrho^n|x' - x''|$, $n \geq 1$, and the verification of (1), (2) is in the framework of Section 3.

5.3. The verification of (3). As in Section 3, the verification of this property is reduced to

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(|X_n| > \varepsilon n^\alpha) = -\infty, \quad \varepsilon > 0. \quad (5.2)$$

In (5.2), we may replace X_n by its copy \tilde{X}_n defined in (5.1). Notice also that

$$|\tilde{X}_n| \leq |A^n x| + \sum_{i=0}^{\infty} \max_{pq} |A_{pq}^i| |\tilde{\xi}|.$$

As was mentioned above, $|A_{pq}^i| \leq K\varrho^i$ for some $K > 0$ and $\varrho \in (0, 1)$. Hence, it suffices to verify

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P\left(\sum_{i=0}^{\infty} \varrho^i |\xi_i| > \varepsilon n^\alpha\right) = -\infty, \quad \varepsilon > 0$$

what be going on similarly to corresponding part of the proof in Section 3.

6. EXOTIC EXAMPLE

Let $(X_n)_{n \geq 0}$, $X_n \in \mathbb{R}$ and $X_0 = x$, be Markov chain defined by the recurrent equation

$$X_n = X_{n-1} - m \frac{X_{n-1}}{|X_{n-1}|} + \xi_n \quad (6.1)$$

where m is a positive parameter, (ξ_n) is i.i.d. sequence of zero mean random variables with

$$Ee^{\delta|\xi_1|} < \infty, \quad \text{for some } \delta > 0,$$

and let $\frac{0}{0} = 0$.

Although the virtual scenario is not completely verifiable here we show that for

$$H(x) = \frac{x}{|x|}$$

the family $(S_n^\alpha)_{n \geq 1}$ possesses the MDP provided that

$$m > \frac{1}{\delta} \log E e^{\delta |\xi_1|}. \quad (6.2)$$

Indeed, by (6.1) we have

$$\frac{1}{n^\alpha} \sum_{k=1}^n \frac{X_{k-1}}{|X_{k-1}|} = \frac{1}{m} \frac{(X_n - x)}{n^\alpha} + \frac{1}{n^\alpha} \sum_{k=1}^n \frac{\xi_k}{m}.$$

The family $(\frac{1}{n^\alpha} \sum_{k=1}^n \frac{\xi_k}{m})_{n \geq 1}$ possesses the MDP with the rate of speed $n^{-(2\alpha-1)}$ and the rate function $I(y) = \frac{m^2}{2E\xi_1^2} y^2$. Then, the family $(S_n^\alpha)_{n \geq 1}$ obeys the same MDP provided that $(\frac{X_n - x}{n^\alpha})_{n \geq 1}$ is exponentially negligible family with the rate $n^{-(2\alpha-1)}$. This verification is reduced to

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(|X_n| > n^\alpha \varepsilon) = -\infty, \quad \varepsilon > 0. \quad (6.3)$$

By the Chernoff inequality $P(|X_n| > n^\alpha \varepsilon) \leq e^{-\delta n^\alpha \varepsilon} E e^{\delta |X_n|}$, that is (6.3) holds if $\sup_{n \geq 1} E e^{\delta |X_n|} < \infty$ for some $\delta > 0$. We show that the latter holds true for δ involved in (6.2). A helpful tool for this verification is the inequality $|z - m \frac{z}{|z|}| \leq ||z| - m|$. Write

$$\begin{aligned} E e^{\delta |X_n|} &= E e^{\delta |X_n|} I(|X_{n-1}| \leq m) + E e^{\delta |X_n|} I(|X_{n-1}| > m) \\ &\leq e^{\delta m} E e^{\delta |\xi_1|} + e^{-\delta m} E e^{\delta |\xi_1|} E e^{\delta |X_{n-1}|}. \end{aligned}$$

Set $\ell = e^{\delta m} E e^{\delta |\xi_1|}$ and $\varrho = e^{-\delta m} E e^{\delta |\xi_1|}$. By (6.2), $\varrho < 1$. Hence, $V(x) = e^{\delta |x|}$ is the Lyapunov function: $P_x V \leq \varrho V(x) + \ell$. Consequently,

$$EV(X_n) \leq \varrho EV(X_n) + \ell, \quad n \geq 1$$

and so, $\sup_{n \geq 1} EV(X_n) \leq V(x) + \frac{\ell}{1-\varrho}$.

APPENDIX A. EXPONENTIALLY INTEGRABLE MARTINGALE-DIFFERENCES

Let $\zeta_n = (\zeta_n)_{n \geq 1}$ be a martingale-difference with respect to some filtration $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$ and $M_n = \sum_{i=1}^n \zeta_i$ be the corresponding martingale.

Theorem A.1. *Assume that for sufficiently small positive δ and any $i \geq 1$*

$$E(e^{\delta |\zeta_i|} | \mathcal{F}_{i-1}) \leq \text{const}. \quad (\text{A.1})$$

Then for any $\alpha \in (0.5, 1)$

$$\lim_{n \rightarrow \infty} \log \frac{1}{n^{2\alpha-1}} P(|M_n| > n\varepsilon) = -\infty.$$

Proof. It suffices to prove $\lim_{n \rightarrow \infty} \frac{1}{n^{2\alpha-1}} \log P(\pm M'_n > n\varepsilon) = -\infty$ and, moreover, it suffices to verify “+” only (“-” is verified similarly).

For fixed positive λ and sufficiently large n , let us introduce the stochastic exponential

$$\mathcal{E}_n(\lambda) = \prod_{i=1}^n E(e^{\lambda \frac{\zeta_i}{n}} | \mathcal{F}_{i-1}).$$

A direct verification shows that

$$E \exp \left(\frac{\lambda M_n}{n} - \log \mathcal{E}_n(\lambda) \right) = 1.$$

We apply this equality for further ones

$$\begin{aligned} 1 &\geq EI \left(M_n > n\varepsilon \right) \exp \left(\frac{\lambda M_n}{n} - \log \mathcal{E}_n(\lambda) \right) \\ &\geq EI \left(M_n > n\varepsilon \right) \exp \left(\lambda\varepsilon - \log \mathcal{E}_n(\lambda) \right). \end{aligned} \quad (\text{A.2})$$

Due to $E(\lambda \frac{\zeta_i}{n} | \mathcal{F}_{i-1}) = 0$ and (A.1), we find that

$$\begin{aligned} \log \mathcal{E}_n(\lambda) &= \sum_{i=1}^n \log \left(1 + E \left[e^{\lambda \frac{\zeta_i}{n}} - 1 - \lambda \frac{\zeta_i}{n} \middle| \mathcal{F}_{i-1} \right] \right) \\ &\leq \sum_{i=1}^n \left\{ \frac{\lambda^2}{2n^2} E((\zeta_i)^2 | \mathcal{F}_{i-1}) + \frac{\lambda^3}{6n^3} E(|\zeta_i|^3 e^{\lambda \frac{|\zeta_i|}{n}} | \mathcal{F}_{i-1}) \right\} \\ &\leq K \left[\frac{\lambda^2}{2n} + \frac{\lambda^3}{6n^2} \right], \end{aligned}$$

where K is some constant. This inequality, being incorporated into (A.2), provides

$$1 \geq EI(M_n > n\varepsilon) \exp \left(\lambda\varepsilon - K \left[\frac{\lambda^2}{2n} + \frac{\lambda^3}{6n^2} \right] \right).$$

If $\varepsilon_0 < 3$, taking $\lambda = \varepsilon_0 n K^{-1}$, we find that

$$\frac{1}{n^{2\alpha-1}} \log P(M_n > n\varepsilon) \leq -\frac{\varepsilon^2 n^{2(1-\alpha)}}{K} \left(\frac{1}{2} - \frac{\varepsilon}{6} \right) \xrightarrow{n \rightarrow \infty} -\infty.$$

Hence, for any verification $\varepsilon > 0$ the desired statement holds true. \square

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